Friction-based scaling of streamwise turbulence intensity in zero-pressure-gradient and pipe flows

Nils T. Basse[a]

[a] *Trubadurens väg 8, 423 41 Torslanda, Sweden*

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Abstract

We explore the analogy between asymptotic scaling of two canonical wall-bounded turbulent flows, i.e. zero-pressure-gradient and pipe flows; we find that these flows can be characterised using similar scaling laws which relate streamwise turbulence intensity and friction.

Keywords:
Streamwise turbulence intensity, friction-based scaling, zero-pressure gradient and pipe flows

A recent paper [1] on zero-pressure-gradient (ZPG) flow has introduced an asymptotic (high Reynolds number) scaling law:

$$\tilde{U_\tau} \sim \frac{1}{\sqrt{\tilde{\delta}}}, \tag{1}$$

where

$$\tilde{U_\tau} = \frac{U_\tau \nu}{M} \sim \frac{\nu}{U_\tau \delta} = \frac{1}{Re_\tau} \tag{2}$$

is named the "dimensionless drag" and

$$\tilde{\delta} = \frac{\delta M}{\nu^2} \sim \frac{\delta^2 U_\tau^2}{\nu^2} = Re_\tau^2 \tag{3}$$

Email address: `nils.basse@npb.dk` (Nils T. Basse)

scales as the friction Reynolds number (Re_τ) squared. Note that we use $\sim$ to mean "scales as". Here, U_τ is the friction velocity, $M = \int_0^\delta U(z)^2 \mathrm{d}z$ is the kinematic momentum rate through the boundary layer, ν is the kinematic viscosity, δ is the boundary layer thickness, z is the distance from the wall and U is the mean velocity in the streamwise direction. Note the asymptotic scaling $M \sim U_\tau^2 \delta$ has been proposed in [1] and applied in Equations (2) and (3).

In this paper we will show that the product $\tilde{U}_\tau \sqrt{\tilde{\delta}}$ scales as the global, i.e. radially averaged, turbulence intensity (TI) $I = \sqrt{\overline{u^2}}/U$, where u is the streamwise velocity fluctuation and overbar denotes time averaging [2, 3, 4]. As a consequence, the squared product ($\tilde{U}_\tau^2 \tilde{\delta}$) scales as the friction factor λ. We note that drag was addressed in [1]; the TI was not discussed.

Our paper represents an expansion of the validity of TI scaling with friction factor, since we have focused exclusively on pipe flow in previous published work. Having a well-defined TI for ZPG flow is important for e.g. computational fluid dynamics (CFD) simulations [5] and this is the main motivation for this work. Another aim is to search for shared features of canonical wall-bounded flows [6] which may lead to improved common models.

The paper is organized as follows: In Section 1, we briefly review results from asymptotic scaling of TI in pipe flows; these findings are related to ZPG flows in Section 2 and we conclude in Section 3.

1. Asymptotic pipe flow scaling of the streamwise turbulence intensity

The material in this section is a summary of research on pipe flow contained in [2, 3, 4]. The local (streamwise) TI is defined as:

$$I_{\mathrm{local}}(r) = \frac{\sqrt{\overline{u^2(r)}}}{U(r)}, \tag{4}$$

where r is the pipe radius ($r = 0$ is the pipe axis and $r = R$ is the pipe wall), i.e. $z = \delta - r = R - r$, which can then be used to define a global TI:

$$I_{\mathrm{global}} = \langle I_{\mathrm{local}}(r) \rangle, \tag{5}$$

where $\langle \cdot \rangle$ indicates radial averaging; see [4], where several definitions of radial averaging have been documented. In the remainder of this paper, we treat the global TI; for simplicity of notation, we drop the subscript "global" and refer to I instead of I_{global}.

For pipe flow, the streamwise turbulence intensity I_{pipe} scales roughly with the ratio of the friction and mean velocities [3, 4]:

$$I_{\text{pipe}} \sim \frac{U_\tau}{U_m} = 2 \times \frac{Re_\tau}{Re_D}, \tag{6}$$

where U_m is the mean velocity and $Re_D = DU_m/\nu$ is the bulk Reynolds number based on the pipe diameter D. For pipe flow, $Re_\tau = RU_\tau/\nu$, where R is the pipe radius. The friction factor λ scales with the square of this ratio:

$$\lambda = 8 \times \frac{U_\tau^2}{U_m^2} = 32 \times \frac{Re_\tau^2}{Re_D^2} \tag{7}$$

As a consequence, the streamwise turbulence intensity scales with the square root of the friction factor:

$$I_{\text{pipe}} \sim \sqrt{\lambda} \tag{8}$$

An example of the scaling using Princeton Superpipe measurements [7, 8] is Equation (23) in [4]:

$$I_{\text{pipe area, AM}} = \frac{1}{R} \int_0^R \frac{\sqrt{\overline{u^2(r)}}}{U(r)} \mathrm{d}r = 0.66 \times \lambda^{0.55}, \tag{9}$$

where AM is an abbreviation for the "arithmetic mean" radial averaging.

2. Equivalence between zero-pressure-gradient and pipe flows

In [9], we have used the "log law" for the streamwise mean velocity [10] to derive a correction term $\sqrt{f_{\text{ZPG}}(Re_\tau)}$ for the asymptotic scaling of drag presented in Equation (1):

$$\tilde{U}_\tau \times \sqrt{f_{\text{ZPG}}(Re_\tau)} = 1.23 \times \tilde{\delta}^{-0.51} \sim \frac{1}{\sqrt{\tilde{\delta}}}, \tag{10}$$

where

$$f_{\mathrm{ZPG}}(Re_\tau) = \frac{2}{\kappa_{\mathrm{ZPG}}^2} - \frac{2A_{\mathrm{ZPG}}}{\kappa_{\mathrm{ZPG}}} + A_{\mathrm{ZPG}}^2 + \log(Re_\tau)\left(\frac{2A_{\mathrm{ZPG}}}{\kappa_{\mathrm{ZPG}}} - \frac{2}{\kappa_{\mathrm{ZPG}}^2}\right) + \log(Re_\tau)^2/\kappa_{\mathrm{ZPG}}^2 \quad (11)$$

Here, $\kappa_{\mathrm{ZPG}} = 0.39$ (von Kármán constant) and $A_{\mathrm{ZPG}} = 5.7$ are constants derived in [9] for a fit to ZPG measurements, see Figure 1.

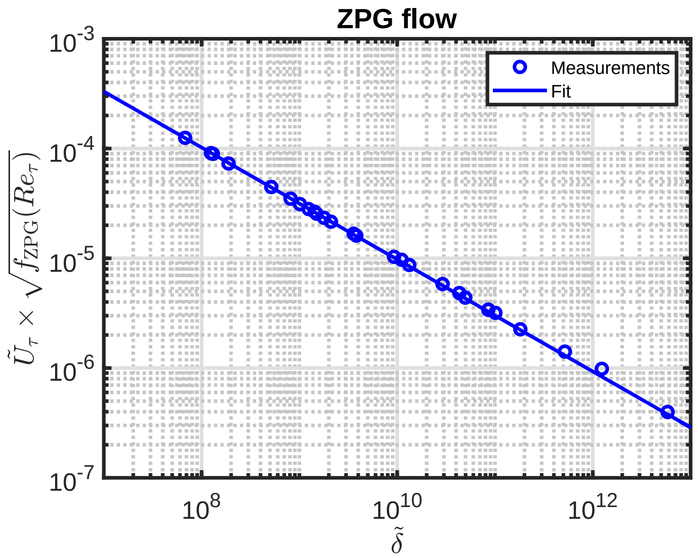

Figure 1: Correction term multiplied by dimensionless drag for ZPG flow as a function of $\tilde{\delta}$. Measurements from [1].

For pipe flow, we can define an equivalent correction term $\sqrt{f_{\mathrm{pipe}}(Re_\tau)}$; above $Re_\tau \sim 11000$ we find $\kappa_{\mathrm{pipe}} = 0.34$ and $A_{\mathrm{pipe}} = 1.0$ [11]. To relate this to the friction factor we perform a fit:

$$\sqrt{f_{\mathrm{pipe}}(Re_\tau)} = 2.24 \times \lambda^{-0.56} \sim \frac{1}{\sqrt{\lambda}} \sim \frac{1}{I_{\mathrm{pipe}}}, \quad (12)$$

see Figure 2. Note that the correction term is different for smooth- and rough-wall flow since A depends on wall roughness [10].

The link between the ZPG and pipe flows is their correction terms, see Figure 3. The correction terms increase monotonically with Reynolds number. To relate the two correction terms, we define their ratio Q and fit this to a logarithmic function:

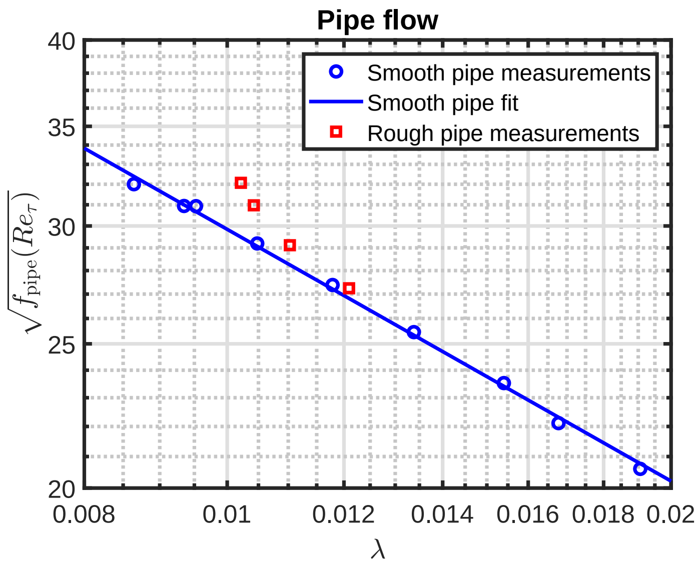

Figure 2: Correction term for pipe flow as a function of λ. Rough pipe measurements are shown for reference. Measurements from [7, 8].

$$Q(Re_\tau) = \frac{\sqrt{f_{\text{pipe}}(Re_\tau)}}{\sqrt{f_{\text{ZPG}}(Re_\tau)}} = 1.15 - 1.46 \times \log(Re_\tau)^{-0.89}, \tag{13}$$

where we note that the constant $1.15 = 0.39/0.34 = \kappa_{\text{ZPG}}/\kappa_{\text{pipe}}$. The ratio approaches an asymptotic value, but the increase towards this value is a rather slow function of Reynolds number.

For ZPG flow, we introduce Equation (10) from [9]:

$$\tilde{U}_\tau = 0.17 \times \tilde{\delta}^{-0.56}, \tag{14}$$

and combine it with Equation (10):

$$\sqrt{f_{\text{ZPG}}(Re_\tau)} = \frac{1.23 \times \tilde{\delta}^{-0.51}}{\tilde{U}_\tau} = 7.24 \times \tilde{\delta}^{0.05} \tag{15}$$

For pipe flow, we combine Equations (9) and (12):

$$\sqrt{f_{\text{pipe}}(Re_\tau)} = 2.24 \times \lambda^{-0.56} = 1.47 \times I_{\text{pipe area, AM}}^{-1.02} \tag{16}$$

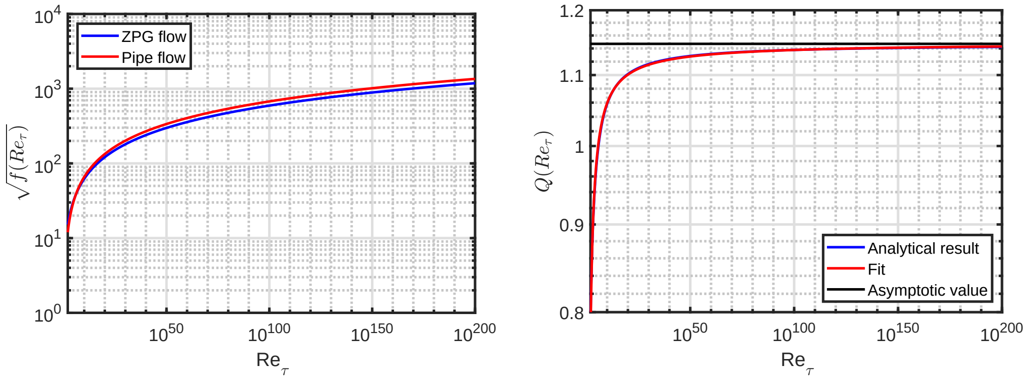

Figure 3: Note the extreme Reynolds number range, from 10^2 to 10^{200}. Left-hand plot: Correction terms for ZPG and pipe flows as a function of Re_τ, right-hand plot: Ratio of correction terms as a function of Re_τ; the horizontal black line indicates the asymptotic value of 1.15.

Since the correction terms in Equations (15) and (16) are related by Equation (13), we arrive at:

$$I_{\text{pipe area, AM}} = 1.19 \times \frac{\tilde{U}_\tau^{0.98}\tilde{\delta}^{0.50}}{Q^{0.98}}, \tag{17}$$

which can be approximated as:

$$I_{\text{pipe}} \sim \frac{\tilde{U}_\tau\sqrt{\tilde{\delta}}}{Q} \tag{18}$$

By using Equations (3) and (14), we can express the product $\tilde{U}_\tau\sqrt{\tilde{\delta}}$ as a function of Re_τ:

$$\tilde{U}_\tau\sqrt{\tilde{\delta}} \sim \tilde{\delta}^{-0.06} \sim Re_\tau^{-0.12}, \tag{19}$$

which is scaling behaviour similar to what has been observed in pipe flow [2, 3, 4]. The exact fit to Equation (19) yields:

$$\tilde{U}_\tau\sqrt{\tilde{\delta}} = 0.10 \times Re_\tau^{-0.11}, \tag{20}$$

see Figure 4.

Based on the findings in this paper we summarise the following TI analogies for ZPG and pipe flows:

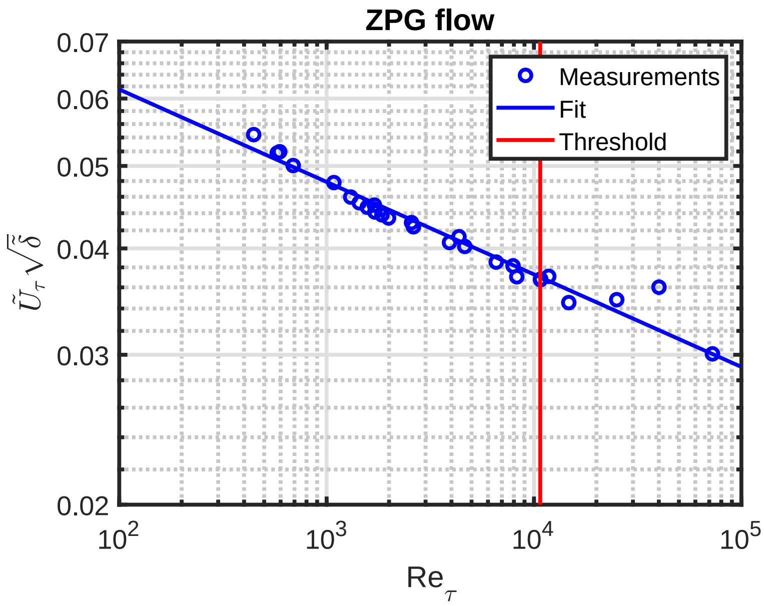

Figure 4: The product $\tilde{U}_\tau\sqrt{\tilde{\delta}}$ as a function of Re_τ. The vertical line at $Re_\tau \sim 11000$ indicates the pipe flow transition found in [11]. Measurements from [1].

$$I_{\text{pipe}} \sim \frac{1}{\sqrt{f_{\text{pipe}}(Re_\tau)}} \sim \frac{\tilde{U}_\tau\sqrt{\tilde{\delta}}}{Q} \sim \frac{I_{\text{ZPG}}}{Q}, \tag{21}$$

where we have used:

$$I_{\text{ZPG}} \sim \frac{1}{\sqrt{f_{\text{ZPG}}(Re_\tau)}} \sim \tilde{U}_\tau\sqrt{\tilde{\delta}} \tag{22}$$

Note that it is only possible to present scaling properties of I_{ZPG} and not an explicit equation, since velocity fluctuation measurements are not available in [1]. We can reformulate Equation (21) to:

$$I_{\text{pipe}}\sqrt{f_{\text{pipe}}} \sim I_{\text{ZPG}}\sqrt{f_{\text{ZPG}}} \tag{23}$$

Corresponding friction factor analogies can be found by taking the square of Equation (21):

$$\lambda \sim \frac{1}{f_{\text{pipe}}(Re_\tau)} \sim \frac{\tilde{U}_\tau^2\tilde{\delta}}{Q^2} \sim \frac{1}{Q^2 f_{\text{ZPG}}(Re_\tau)} \tag{24}$$

3. Conclusions

We have explored the correspondence between zero-pressure-gradient (ZPG) and pipe flows for asymptotic scaling of streamwise turbulence intensity with friction. It is demonstrated that similar scalings are valid for both types of flows; the product $\tilde{U}_\tau \sqrt{\tilde{\delta}}$ for ZPG flow is equivalent to the streamwise turbulence intensity for pipe flow I_{pipe}. The scaling of turbulence intensity with Reynolds number in ZPG flow closely matches the corresponding pipe flow scaling. In addition, we have shown that the turbulence intensity is inversely proportional to the correction term $\sqrt{f(Re_\tau)}$ and that κ and A for the correction term are different for ZPG and pipe flows.

A source of inaccuracy of our results is that we have used measurements carried out at all Reynolds numbers. However, in [11] we have shown that a transition exists at $Re_\tau \sim 11000$ for pipe flow; scaling is somewhat different below and above this threshold. Future research includes studies for higher Reynolds numbers to characterise scaling below and above the transition. We also plan studies of other canonical flows, e.g. channel flow [6] and plane Couette and Poiseuille flows [12].

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Data availability statement. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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